Logical Foundations of Categorization Theory

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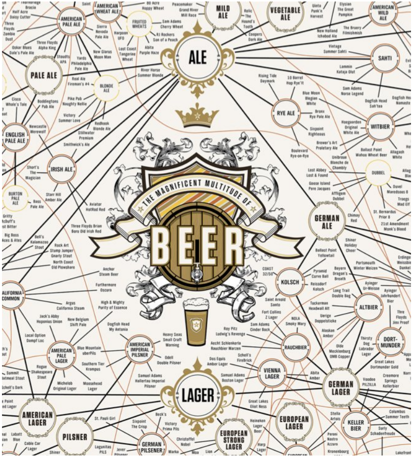
What is categorization?

From Wikipedia:
Categorization is the process in which ideas and objects are recognized, differentiated, and understood.
Ideally, a category illuminates a relationship between the subjects and objects of knowledge.
Categorization is fundamental in language, prediction, inference, decision-making and in all kinds of environmental interaction.



Overview and General Motivation

- **Truly interdisciplinary:** philosophy, cognition, social/management science, linguistics, AI.
- rapid development, **different approaches**;
- emerging **unifying perspective:** categories are dynamic in their essence; they shape and are shaped by processes of social interaction.
- **Data-driven** developments, both empirical and theoretical.
- However, what is **lacking**:
  - a **common ground** for the various approaches;
  - formal models addressing **dynamics** and connections with the processes of **social interaction**.
- **Research program:** logic as common ground; dynamics as starting point rather than outcome; systematic connection between dynamics and processes of social interaction.
- This involves **exploring seriously uncharted territory** in the mathematical theory of nonclassical logics.

How did I get interested in categories?

Uniform duality-theoretic approaches to nonclassical logics

- canonical extensions;
- unified correspondence;
- updates on algebras;
- multi-type calculi;
- multi-type **algebraic proof theory**.

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Mathematical theory of LE-logics (LE: lattice expansions)

- algebraic and state-based (aka relational) semantics;
- generalized Sahlqvist correspondence and canonicity;
- syntactic and semantic cut elimination, finite model property;
- Goldblatt-Thomason theorem.

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Can we make intuitive sense of LE-logics?

Basic lattice logic & main ideas

Language: $\mathcal{L} \ni \phi ::= p \in Prop \mid \top \mid \bot \mid \phi \wedge \phi \mid \phi \vee \phi$

Lattice Logic: Set of $\mathcal{L}$-sequents $\phi \vdash \psi$

- containing:

  $p \vdash p \quad \bot \vdash p \quad p \vdash \top \quad p \vdash p \vee q \quad q \vdash p \vee q \quad p \wedge q \vdash p \quad p \wedge q \vdash q$

- closed under:

  $\frac{\phi \vdash \chi \quad \chi \vdash \psi}{\phi \vdash \psi} \quad \frac{\phi \vdash \psi}{\phi(\chi/p) \vdash \psi(\chi/p)} \quad \frac{\chi \vdash \phi \quad \chi \vdash \psi}{\chi \vdash \phi \wedge \psi} \quad \frac{\phi \vdash \chi \quad \psi \vdash \chi}{\phi \vee \psi \vdash \chi}$

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Challenge: Interpreting $\vee$ as 'or' and $\wedge$ as 'and' does not work, since 'and' and 'or' distribute over each other, while $\wedge$ and $\vee$ don't.

Proposal: Interpreting $\phi \in \mathcal{L}$ as **other entities than sentences**?

Examples: categories, concepts, theories, interrogative agendas.

The interpretation of $\vee$ and $\wedge$ in all these contexts is ok with failure of distributivity!

Approach:

- Understand LE-logics as the logics of **these entities**;
- integrate LE-logics into more expressive logics capturing how these entities **interact** (e.g. with sentences, actions etc.).

LE-logics as the logics of formal concepts

Birkhoff (1938): "lattice theory provides the proper vocabulary for discussing order, and especially systems which are in any sense **hierarchies**."

Wille (1973-): "Formal Concept Analysis has been developed as a subfield of Applied Mathematics based on the mathematization of concept and concept hierarchy."

"Concepts can be philosophically understood as the basic units of thought [...]. According to the main philosophical tradition, a concept is constituted by its *extension*, comprising all *objects* which belong to the concept, and its *intension*, including all *attributes* which apply to all objects of the extension [...]. Concepts can only live in relationships with many other concepts where the subconcept-superconcept-relation plays a prominent role."

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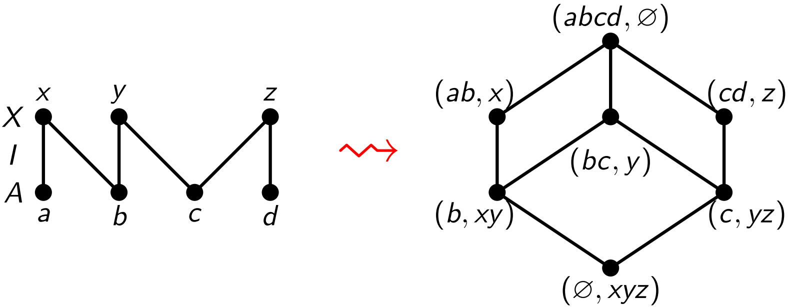
Formal contexts as models of lattice logic



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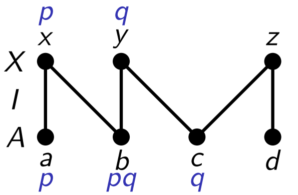
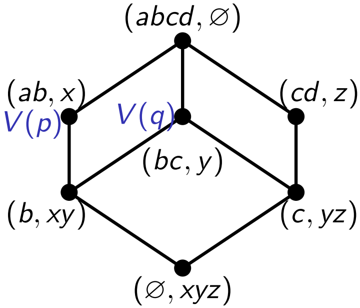
- closed under:
  $\frac{\phi\vdash\chi \quad \chi\vdash\psi}{\phi\vdash\psi} \qquad \frac{\phi\vdash\psi}{\phi(\chi/p)\vdash\psi(\chi/p)} \qquad \frac{\chi\vdash\phi \quad \chi\vdash\psi}{\chi\vdash\phi\wedge\psi} \qquad \frac{\phi\vdash\chi \quad \psi\vdash\chi}{\phi\vee\psi\vdash\chi}$

Formal contexts as models of lattice logic



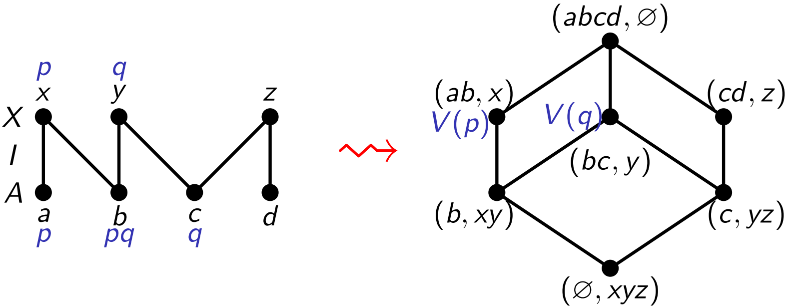


Let $\mathbb{P} = (A, X, I)$ and $\mathbb{P}^+$ be the complex algebra of $\mathbb{P}$.
Models: $\mathbb{M} := (\mathbb{P}, V)$ with $V : Prop \to \mathbb{P}^+$

$$V(p) := ([\![p]\!], (\![p]\!))$$

membership: $\mathbb{M}, a \Vdash p$ iff $a \in [\![p]\!]_{\mathbb{M}}$
description: $\mathbb{M}, x \succ p$ iff $x \in (\![p]\!)_{\mathbb{M}}$

Formal contexts as models of lattice logic



$\mathbb{M}, a \Vdash \bot$ iff $\forall x(aIx)$

$\mathbb{M}, a \Vdash \top$ always

$\mathbb{M}, x \succ \bot$ always

$\mathbb{M}, x \succ \top$ iff $\forall a(aIx)$

$\mathbb{M}, a \Vdash \phi \wedge \psi$ iff $\mathbb{M}, a \Vdash \phi$ and $\mathbb{M}, a \Vdash \psi$

$\mathbb{M}, x \succ \phi \wedge \psi$ iff for all $a \in A$, if $\mathbb{M}, a \Vdash \phi \wedge \psi$, then aIx

$\mathbb{M}, a \Vdash \phi \vee \psi$ iff for all $x \in X$, if $\mathbb{M}, x \succ \phi \vee \psi$, then aIx

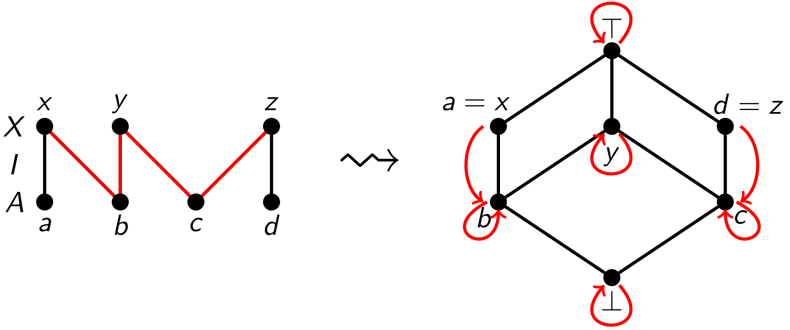
$\mathbb{M}, x \succ \phi \vee \psi$ iff $\mathbb{M}, x \succ \phi$ and $\mathbb{M}, x \succ \psi$

$$\mathbb{M} \models \phi \vdash \psi \text{ iff } [\![\phi]\!] \subseteq [\![\psi]\!] \text{ iff } (\![\psi]\!) \subseteq (\![\phi]\!)$$

Expanding the language with modal operators

Enriched formal contexts: $\mathbb{F} = (A, X, I, \{R_i \mid i \in Agents\})$

$R_i \subseteq A \times X$ and $\forall a((R^{\uparrow}[a])^{\downarrow\uparrow} = R^{\uparrow}[a])$ and $\forall x((R^{\downarrow}[x])^{\uparrow\downarrow} = R^{\downarrow}[x])$



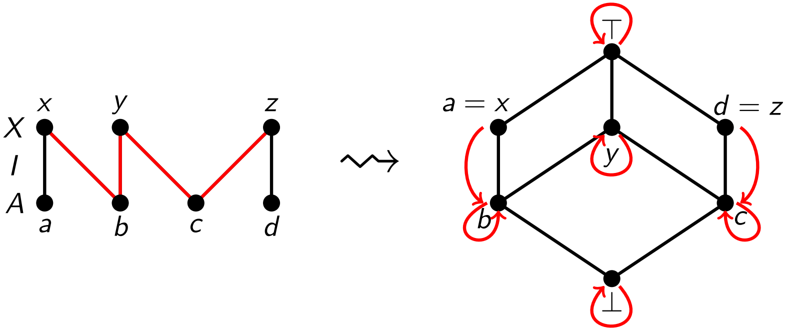
Language: $\mathcal{L}' \ni \phi ::= p \in Prop \mid \top \mid \bot \mid \phi \wedge \phi \mid \phi \vee \phi \mid \Box_i \phi$

$\Box_i \phi$: concept ϕ **according to** agent i

Logic:

- Additional axioms: $\top \vdash \Box_i \top \quad \Box_i \phi \wedge \Box_i \psi \vdash \Box_i(\phi \wedge \psi)$
- Additional rule: $\frac{\phi \vdash \psi}{\Box_i \phi \vdash \Box_i \psi}$

Interpretation of $\Box_i$-formulas on enriched formal contexts



$$V(\Box_i \phi) = \Box_i V(\phi) = (R_i^{\downarrow}[([\phi])], (R_i^{\downarrow}[([\phi])])^{\uparrow})$$

$\mathbb{M}, a \Vdash \Box_i \phi$ iff for all $x \in X$, if $\mathbb{M}, x \succ \phi$, then aR_ix

$\mathbb{M}, x \succ \Box_i \phi$ iff for all $a \in A$, if $\mathbb{M}, a \Vdash \Box_i \phi$, then aIx

Epistemic interpretation of modal axioms

Reflexivity aka Factivity

$\forall p(\Box_i p \leq p)$ iff $R_i \subseteq I$

Agent i's attributions are factually correct!

Symmetry

$\forall p(p \leq \Box_i \Diamond_i p)$ iff $R_{\Diamond_i} \subseteq R_{\blacklozenge_i}$

If agent i recognizes feature x as an a-feature, then i must also recognize object a as an x-object.

Transitivity aka Positive introspection

$\forall p(\Box_i p \leq \Box_i \Box_i p)$ iff $R_i \subseteq R_i ; R_i$

If agent i recognizes object a as an x-object, then i must also attribute to a all the features shared by x-objects according to i.

Non epistemic interpretation: rough concepts

Conceptual approximation spaces: $\mathbb{F} = (A, X, I, R_\Box, R_\Diamond)$ with $R_\Box \subseteq A \times X$ and $R_\Diamond \subseteq X \times A$, I-compatible and s.t. $R_\blacksquare \,;\, R_\Box \subseteq I$.

Fact: $\mathbb{F} \models \Box p \vdash \Diamond p$ iff $R_\blacksquare \,;\, R_\Box \subseteq I$

$$
\begin{array}{lll}
\mathbb{M}, a \Vdash \Box\varphi & \text{iff} & \text{for all } x \in X, \text{ if } \mathbb{M}, x \succ \varphi, \text{ then } aR_\Box x \\
\mathbb{M}, x \succ \Box\varphi & \text{iff} & \text{for all } a \in A, \text{ if } \mathbb{M}, a \Vdash \Box(\varphi), \text{ then } aIx, \\
\mathbb{M}, a \Vdash \Diamond\phi & \text{iff} & \text{for all } x \in X, \text{ if } \mathbb{M}, x \succ \Diamond\phi, \text{ then } aIx \\
\mathbb{M}, x \succ \Diamond\phi & \text{iff} & \text{for all } a \in A, \text{ if } \mathbb{M}, a \Vdash \phi, \text{ then } aR_\Diamond x.
\end{array}
$$

If (A, X, I) database and $R \subseteq A \times X$ I-compatible,

aIx stands for "object a has feature x"
aRx stands for "object a demonstrably has feature x"

If $R_\Box := R$ and $R_\Diamond := R^{-1}$, then

$$[\![\Box\phi]\!] = \{a \in A \mid \forall x(x \succ \phi \Rightarrow aRx)\}$$ **certified members** of ϕ.

$$(\![\Diamond\phi]\!) = \{x \in X \mid \forall a(a \Vdash \phi \Rightarrow aRx)\},$$

hence $[\![\Diamond\phi]\!] :=$ **possible members** of ϕ.

Rough concepts: unifying Rough Set Theory and FCA

Conceptual approximation spaces

polarity-based frames $\mathbb{F} = (\mathbb{P}, R_\Box, R_\Diamond)$ s.t.:

$$R_\Box ; R_\blacksquare \subseteq I \tag{1}$$

$\mathbb{F}$ is

- *reflexive* if $R_\Box \subseteq I$ and $R_\blacksquare \subseteq I$;
- *symmetric* if $R_\Diamond = R$;
- *transitive* if $R_\Box \subseteq R_\Box ; R_\Box$ and $R_\Diamond \subseteq R_\Diamond ; R_\Diamond$.

LE-$\mathcal{ALC}$: unifying Description Logic and FCA

- Disjoint sets OBJ and FEAT (**individual names** for objects and features);
- **Role names** for LE-$\mathcal{ALC}$:
  $I \subseteq \mathrm{OBJ} \times \mathrm{FEAT} \quad R_\Box \subseteq \mathrm{OBJ} \times \mathrm{FEAT} \quad R_\Diamond \subseteq \mathrm{FEAT} \times \mathrm{OBJ}$
- **Language** of LE-$\mathcal{ALC}$-concepts:
  $$C ::= D \mid C_1 \wedge C_2 \mid C_1 \vee C_2 \mid \top \mid \bot \mid \langle R_\Diamond \rangle C \mid [R_\Box] C$$
  where $D \in \mathcal{C}$, given set of atomic concept names;
- **TBox assertions** : $C_1 \equiv C_2$ ($C_1 \sqsubseteq C_2$ defined as $C_1 \equiv C_2 \wedge C_3$, for some fresh concept name C_3)
- **ABox assertions**:
  $$aR_\Box x, \quad xR_\Diamond a, \quad aIx, \quad a : C, \quad x :: C, \quad \neg\alpha,$$

Models of LE-$\mathcal{ALC}$

Tuples $\mathrm{M} = (\mathbb{F}, \cdot^{\mathrm{I}})$, s.t.

- $\mathbb{F} = (A, X, I, R_{\Box}, R_{\Diamond})$ enriched formal context;
- $\cdot^{\mathrm{I}}$ interpretation map:
  - $a^{\mathrm{I}} \in A$ and $x^{\mathrm{I}} \in X$ for each $a \in$ OBJ and $x \in$ FEAT;
  - $D^{\mathrm{I}} \in \mathbb{F}^{+}$ for each $D \in \mathcal{C}$

Complex concept names are interpreted homomorphically:

$$\bot^{\mathrm{I}} = (X^{\downarrow}, X) \qquad \top^{\mathrm{I}} = (A, A^{\uparrow}) \qquad (C_1 \wedge C_2)^{\mathrm{I}} = C_1^{\mathrm{I}} \wedge C_2^{\mathrm{I}}$$
$$(C_1 \vee C_2)^{\mathrm{I}} = C_1^{\mathrm{I}} \vee C_2^{\mathrm{I}} \qquad ([R_{\Box}]C)^{\mathrm{I}} = [R_{\Box}]C^{\mathrm{I}} \qquad (\langle R_{\Diamond}\rangle C)^{\mathrm{I}} = \langle R_{\Diamond}\rangle C^{\mathrm{I}}$$

Results:

- **Tableaux algorithm** for consistency-checking of LE-$\mathcal{ALC}$ knowledge bases with acyclic TBoxes;
- proof of **termination, soundness**, and **completeness**.
- **Complexity**: PTIME-complete, while analogous problem for $\mathcal{ALC}$ is PSPACE-complete.

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