On the φ-degree of inclusion

Manuel Ojeda-Aciego

Departamento Matemática Aplicada

Universidad de Málaga (Spain)

(mainly based on joint work with Nicolás Madrid)

July 20, 2023
ICFCA, Kassel

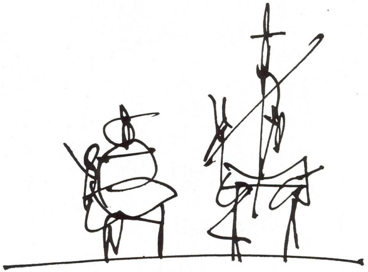
Presentación

(Sí, en español)

Por ver que tiene este caso un no se qué ... yo, por mi parte, os oiré, hermano, de muy buena gana, y así lo harán todos estos señores, por lo mucho que tienen de discretos y de ser amigos de curiosas novedades que suspendan, alegren y entretengan los sentidos, como, sin duda, pienso que lo ha de hacer vuestro cuento.

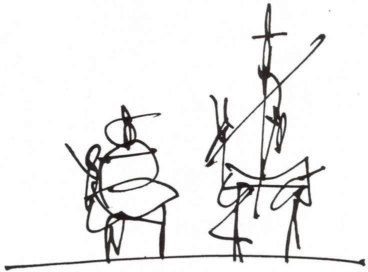
Comenzad, pues, amigo,
que todos escucharemos.

[[Quijote I, cap 50]]



El Quijote

Opening
(in English)



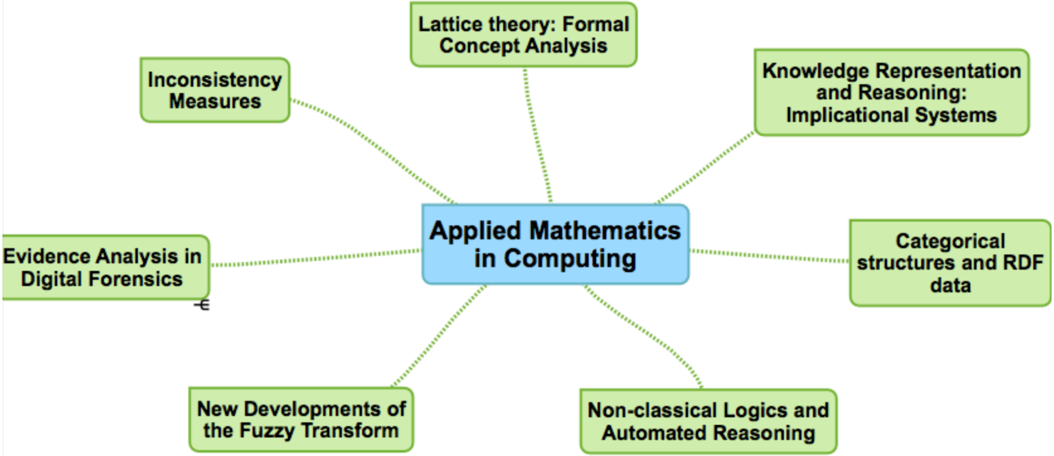
El Quijote

Seeing that this affair has a certain colour of chivalry about it, I for my part, brother, will hear you most gladly, and so will all these gentlemen, from the high intelligence they possess and their love of curious novelties that interest, charm, and entertain the mind, as I feel quite sure your story will do.

So begin, friend, for we are all prepared to listen.

[[Quixote I, chap 50]]

Recent research lines of our group

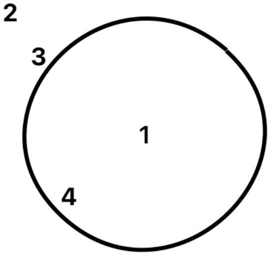


Introduction

- The notion of inclusion is undoubtedly one of the most important concepts in set theory.
- There is currently no consensus on how to extend this notion to fuzzy sets.

L.A. Zadeh (1965). Fuzzy sets. *Information and Control* 8(3):338–353

- A standard (crisp) set $\boldsymbol{A}$ can be identified with its characteristic function $\chi_{\boldsymbol{A}} \colon \mathcal{U} \to \{0, 1\}$.

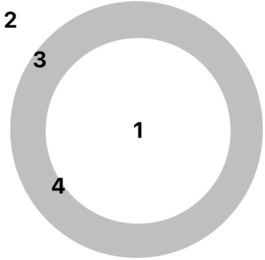


Crisp set A

$$\mathcal{U} = \{1, 2, 3, 4\}$$

$$\chi_{\boldsymbol{A}}(\boldsymbol{u}) = \begin{cases} 1 & \text{if } \boldsymbol{u} = 1 \\ 0 & \text{if } \boldsymbol{u} = 2 \\ 0 & \text{if } \boldsymbol{u} = 3 \\ 1 & \text{if } \boldsymbol{u} = 4 \end{cases}$$

- A fuzzy set $\boldsymbol{A}$ can be identified with its characteristic function $\chi_{A}\colon \mathcal{U} \to [0,1]$.

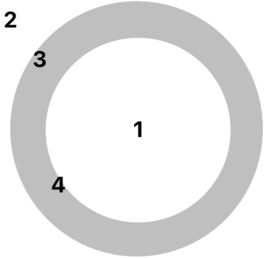


Fuzzy set B

$$\mathcal{U} = \{1, 2, 3, 4\}$$

$$\chi_A(u) = \begin{cases} 1 & \text{if } u = 1 \\ 0 & \text{if } u = 2 \\ 0.2 & \text{if } u = 3 \\ 0.9 & \text{if } u = 4 \end{cases}$$

- A fuzzy set $\boldsymbol{A}$ can be identified with its characteristic function $\chi_{\boldsymbol{A}} \colon \mathcal{U} \to [0,1]$.



Fuzzy set B

$$\mathcal{U} = \{1, 2, 3, 4\}$$

$$\chi_{\boldsymbol{A}}(\boldsymbol{u}) = \begin{cases} 1 & \text{if } \boldsymbol{u} = 1 \\ 0 & \text{if } \boldsymbol{u} = 2 \\ 0.2 & \text{if } \boldsymbol{u} = 3 \\ 0.9 & \text{if } \boldsymbol{u} = 4 \end{cases}$$

- It is usual to abuse notation and write $\boldsymbol{A}(\boldsymbol{u})$ instead of $\chi_{\boldsymbol{A}}(\boldsymbol{u})$.

How to define the inclusion between fuzzy sets?

There are, at least, three approaches:

- The "analytic" one
- The "axiomatic" one
- The "functional" one

Introduction

How to define the inclusion between fuzzy sets?

There are, at least, three approaches:

- The "analytic" one, in which we have
  - A *crisp* approach to inclusion between fuzzy sets

$$A \subseteq B \iff A(u) \leq B(u) \quad \forall u \in \mathcal{U}$$

  - A *gradual* approach to inclusion between fuzzy sets

$$S(A, B) = \bigwedge_{u \in \mathcal{U}} A(u) \to B(u),$$

L.A. Zadeh (1965). Fuzzy sets. *Information and Control* 8(3):338–353

J.A. Goguen (1967). ***L***-fuzzy sets. *J. of Mathematical Analysis and Applications* 18:145–174

Introduction

How to define the inclusion between fuzzy sets?

There are, at least, three approaches:

- The "axiomatic" one, for instance the approaches by
  - Sinha & Dougherty
  - Kitainik
  - Young

D. Sinha and E.R. Dougherty (1993). Fuzzification of set inclusion: Theory and applications. *Fuzzy Sets and Systems*, 55(1):15–42

L.M. Kitainik (1987). Fuzzy Inclusions and Fuzzy Dichotomous Decision Procedures. In *Optimization Models Using Fuzzy Sets and Possibility Theory*, pages 154–170, Springer

V. Young (1996). Fuzzy subsethood. *Fuzzy Sets and Systems*, 77(3):371–384

Introduction

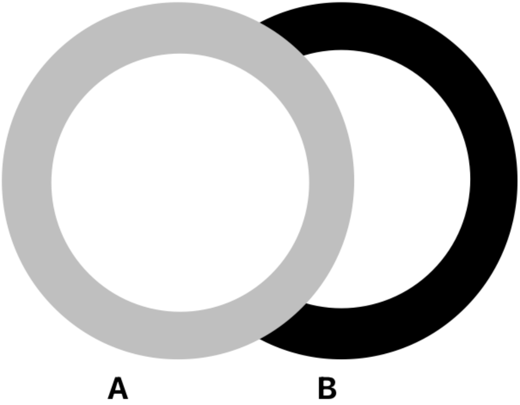
How to define the inclusion between fuzzy sets?

There are, at least, three approaches:

- The "functional" one, based on $\boldsymbol{f}$-inclusion

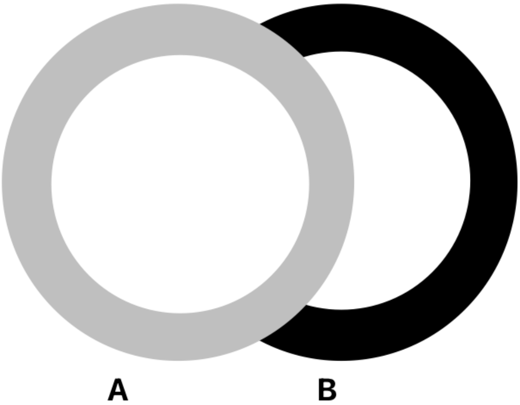
N. Madrid, **MOA**, and I. Perfilieva. *f-inclusion indexes between fuzzy sets*. In Proc. of IFSA-EUSFLAT, 2015

f-index of inclusion



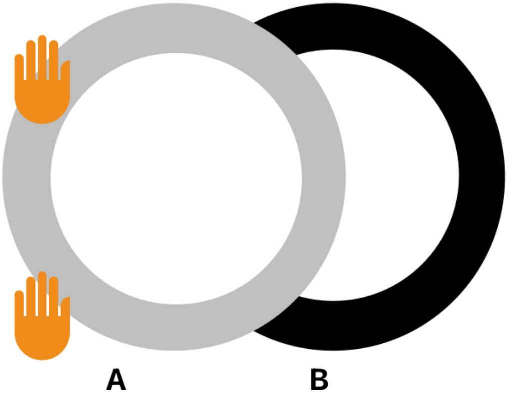
f-index of inclusion

Intuitive idea: squeeze A to fit into B.



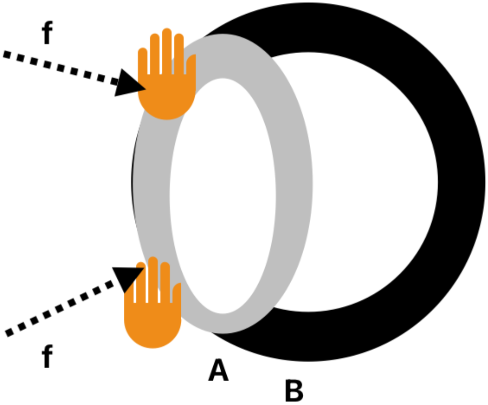
f-index of inclusion

Intuitive idea: squeeze A to fit into B.



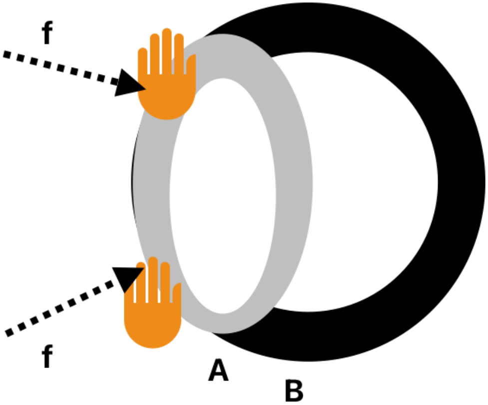
f-index of inclusion

Intuitive idea: squeeze A to fit into B.



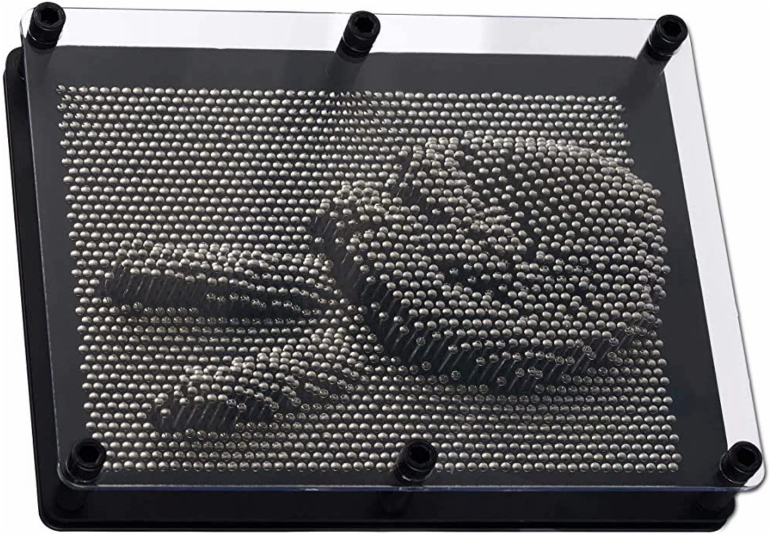
f-index of inclusion

Intuitive idea: squeeze A to fit into B. Voilà, here we have an f . . .



f-index of inclusion

A more realistic interpretation



f-index of inclusion

The f-index of inclusion will be defined in three steps.

Definition 1

The set of f-indexes of inclusion, denoted Ω, is the set of all the increasing mappings $f: [0,1] \to [0,1]$ satisfying $f(x) \leq x$ for all $x \in [0,1]$.

f-index of inclusion

The f-index of inclusion will be defined in three steps.

Definition 1

The set of f-indexes of inclusion, denoted Ω, is the set of all the increasing mappings $f\colon [0,1] \to [0,1]$ satisfying $f(x) \le x$ for all $x \in [0,1]$.

Definition 2

Let A and B be fuzzy sets and consider $f \in \Omega$. We say that A *is f-included in* B (denoted $A \subseteq_f B$) if and only if $f(A(u)) \le B(u)$ holds for all $u \in \mathcal{U}$.

f-index of inclusion

Definition 3

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be fuzzy sets, the $\boldsymbol{f}$*-index of inclusion* of $\boldsymbol{A}$ in $\boldsymbol{B}$, denoted $\mathbf{Inc}(\boldsymbol{A}, \boldsymbol{B})$, is defined as

$$\mathbf{Inc}(A, B) = \mathbf{max}\{f \in \Omega \mid A \subseteq_f B\}$$

f-index of inclusion

Definition 3

Let A and B be fuzzy sets, the f*-index of inclusion* of A in B, denoted $\mathbf{Inc}(A, B)$, is defined as

$$\mathbf{Inc}(A, B) = \max\{f \in \Omega \mid A \subseteq_f B\}$$

Theorem

Let A and B be two fuzzy sets, then $\mathbf{Inc}(A, B)(x) = x \wedge \bigwedge_{u \in \mathcal{U}} \{B(u) \mid x \leq A(u)\}.$

f-index of inclusion

Definition 3

Let A and B be fuzzy sets, the f*-index of inclusion* of A in B, denoted $\mathbf{Inc}(A, B)$, is defined as

$$\mathbf{Inc}(A, B) = \max\{f \in \Omega \mid A \subseteq_f B\}$$

Theorem

Let A and B be two fuzzy sets, then $\mathbf{Inc}(A, B)(x) = x \wedge \bigwedge_{u \in \mathcal{U}} \{B(u) \mid x \leq A(u)\}$.

In order to justify the use of $\mathbf{Inc}(A, B)$ as a suitable index of inclusion between two fuzzy sets, we recall the axiomatic approach by Sinha & Dougherty.

f-index of inclusion

Definition 3

Let A and B be fuzzy sets, the *f-index of inclusion* of A in B, denoted $\mathbf{Inc}(A, B)$, is defined as

$$\mathbf{Inc}(A, B) = \max\{f \in \Omega \mid A \subseteq_f B\}$$

Theorem

Let A and B be two fuzzy sets, then $\mathbf{Inc}(A, B)(x) = x \wedge \bigwedge_{u \in \mathcal{U}} \{B(u) \mid x \leq A(u)\}$.

In order to justify the use of $\mathbf{Inc}(A, B)$ as a suitable index of inclusion between two fuzzy sets, we recall the axiomatic approach by Sinha & Dougherty.

N. Madrid and **MOA**. *Functional degrees of inclusion and similarity between L-fuzzy sets*. Fuzzy Sets and Systems, **390** (2020), 1–22.

Axioms of Sinha & Dougherty

Definition

The mapping $S\colon \mathcal{F}(\mathcal{U}) \times \mathcal{F}(\mathcal{U}) \to [0,1]$ is an *SD-measure of inclusion* if the following axioms hold for all fuzzy sets A, B and C:

(SD1) $S(A,B) = 1$ if and only if $A(u) \leq B(u)$ for all $u \in \mathcal{U}$.

(SD2) $S(A,B) = 0$ if and only if there is $u \in \mathcal{U}$ such that $A(u) = 1$ and $B(u) = 0$.

(SD3) Si $B(u) \leq C(u)$ for all $u \in \mathcal{U}$ then $S(A,B) \leq S(A,C)$.

(SD4) Si $B(u) \leq C(u)$ for all $u \in \mathcal{U}$ then $S(C,A) \leq S(B,A)$.

(SD5) If $T\colon \mathcal{U} \to \mathcal{U}$ is a bijective mapping in $\mathcal{U}$, then $S(A,B) = S(T(A), T(B))$.

(SD6) $S(A,B) = S(B^c, A^c)$.

(SD7) $S(A \cup B, C) = \min\{S(A,C), S(B,C)\}$.

(SD8) $S(A, B \cap C) = \min\{S(A,B), S(A,C)\}$.

Theorem

Given fuzzy sets A, B *and* C

1. $\mathrm{Inc}(A, B) = id$ *if and only if* $A(u) \leq B(u)$ *for all* $u \in \mathcal{U}$.
2. $\mathrm{Inc}(A, B) = \bot$ *if and only if there exists a sequence* $\{u_i\}_{i\in\mathbb{N}} \subseteq \mathcal{U}$ *such that* $A(u_i) = 1$ *for all* $i \in \mathbb{N}$ *and* $\bigwedge_{i\in\mathbb{N}} B(u_i) = 0$.
3. *If* $B(u) \leq C(u)$ *for all* $u \in \mathcal{U}$ *then* $\mathrm{Inc}(C, A) \leq \mathrm{Inc}(B, A)$.
4. *If* $B(u) \leq C(u)$ *for all* $u \in \mathcal{U}$ *then* $\mathrm{Inc}(A, B) \leq \mathrm{Inc}(A, C)$;
5. *Let* $T : \mathcal{U} \to \mathcal{U}$ *be bijective on* $\mathcal{U}$, *then* $\mathrm{Inc}(A, B) = \mathrm{Inc}(T(A), T(B))$.
6.
7. $\mathrm{Inc}(A, B \cap C) = \mathrm{Inc}(A, B) \wedge \mathrm{Inc}(A, C)$.
8. $\mathrm{Inc}(A \cup B, C) = \mathrm{Inc}(A, C) \wedge \mathrm{Inc}(B, C)$.

f-index of inclusion (comments)

- Notice that (SD6) does not appear in the theorem.

Definition

Let A and B be fuzzy sets, the *measure of inclusion* of A in B is defined by

$$M_{\mathrm{Inc}}(A, B) = 2\int_0^1 \mathrm{Inc}(A, B)(x)dx.$$

Theorem (SD6 recovered)

Let A and B be fuzzy sets, then $M_{\mathrm{Inc}}(A, B) = M_{\mathrm{Inc}}(B^c, A^c)$.

f-index of inclusion (comments)

- Notice that (SD6) does not appear in the theorem.

Definition

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be fuzzy sets, the *measure of inclusion* of $\boldsymbol{A}$ in $\boldsymbol{B}$ is defined by

$$M_{\text{Inc}}(A, B) = 2\int_0^1 \text{Inc}(A, B)(x)dx.$$

Theorem (SD6 recovered)

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be fuzzy sets, then $M_{\text{Inc}}(A, B) = M_{\text{Inc}}(B^c, A^c)$.

f-index of inclusion (comments)

- Notice that (SD6) does not appear in the theorem.

Definition

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be fuzzy sets, the *measure of inclusion* of $\boldsymbol{A}$ in $\boldsymbol{B}$ is defined by

$$M_{\mathrm{Inc}}(A, B) = 2\int_0^1 \mathrm{Inc}(A, B)(x)dx.$$

Theorem (SD6 recovered)

Let $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *be fuzzy sets, then* $M_{\mathrm{Inc}}(A, B) = M_{\mathrm{Inc}}(B^c, A^c)$.

Applications of f-inclusion

f-index of inclusion and entropy

Definition (De Luca & Termini, 1972)

A mapping $\boldsymbol{E}\colon \mathcal{F}(\mathcal{U}) \to \mathbb{R}$ is called a *measure of fuzzy entropy* if it satisfies the following axioms for all fuzzy sets $\boldsymbol{A}$ and $\boldsymbol{B}$:

(E1) $\boldsymbol{E}(\boldsymbol{A}) = \boldsymbol{0}$ if and only if $\boldsymbol{A}(\boldsymbol{u}) \in \{0, 1\}$ for all $\boldsymbol{u} \in \mathcal{U}$.

(E2) $\boldsymbol{E}(\boldsymbol{A})$ reaches the maximum value if and only if $\boldsymbol{A}(\boldsymbol{u}) = \boldsymbol{0.5}$ for all $\boldsymbol{u} \in \mathcal{U}$.

(E3) $\boldsymbol{E}(\boldsymbol{A}) \leq \boldsymbol{E}(\boldsymbol{B})$ if $\boldsymbol{A}$ is a refinement of $\boldsymbol{B}$.

A. De Luca and S. Termini. *A definition of a non-probabilistic entropy in the setting of fuzzy sets theory*. Information and Control, 20(4):301–312, 1972.

Definition (De Luca & Termini, 1972)

A mapping $\boldsymbol{E}\colon \mathcal{F}(\mathcal{U}) \to \mathbb{R}$ is called a *measure of fuzzy entropy* if it satisfies the following axioms for all fuzzy sets $\boldsymbol{A}$ and $\boldsymbol{B}$:

(E1) $\boldsymbol{E}(\boldsymbol{A}) = \mathbf{0}$ if and only if $\boldsymbol{A}(\boldsymbol{u}) \in \{0, 1\}$ for all $\boldsymbol{u} \in \mathcal{U}$.

(E2) $\boldsymbol{E}(\boldsymbol{A})$ reaches the maximum value if and only if $\boldsymbol{A}(\boldsymbol{u}) = \mathbf{0.5}$ for all $\boldsymbol{u} \in \mathcal{U}$.

(E3) $\boldsymbol{E}(\boldsymbol{A}) \leq \boldsymbol{E}(\boldsymbol{B})$ if $\boldsymbol{A}$ is a refinement of $\boldsymbol{B}$ (that is, if $\boldsymbol{A}(\boldsymbol{u}) \leq \boldsymbol{B}(\boldsymbol{u})$ if $\boldsymbol{B}(\boldsymbol{u}) \leq \mathbf{0.5}$ and $\boldsymbol{B}(\boldsymbol{u}) \leq \boldsymbol{A}(\boldsymbol{u})$ if $\boldsymbol{B}(\boldsymbol{u}) \geq \mathbf{0.5}$).

A. De Luca and S. Termini. *A definition of a non-probabilistic entropy in the setting of fuzzy sets theory*. Information and Control, 20(4):301–312, 1972.

Definition

Let $\boldsymbol{A}$ be a fuzzy set defined on a finite universe, and let $\boldsymbol{n}$ be a strictly decreasing negation such that $\boldsymbol{n(0.5) = 0.5}$. The measure of fuzzy entropy $\boldsymbol{E}$ induced by the φ-index of inclusion and the negation $\boldsymbol{n}$ is given by:

$$\boldsymbol{E(A) = \widehat{M}_{Inc}(A \cup n(A), A \cap n(A))}$$

where $\boldsymbol{\widehat{M}_{Inc}}$ is defined by

$$\boldsymbol{\widehat{M}_{Inc}(A, B) = \frac{\sum_{u \in \mathcal{U}} M_{Inc}(A_u, B_u)}{Card(\mathcal{U})}}$$

Theorem

The mapping $\boldsymbol{E}$ induced by the φ-index of inclusion and a strictly decreasing negation $\boldsymbol{n}$ such that $\boldsymbol{n(0.5) = 0.5}$, is a De Luca-Termini measure of fuzzy entropy.

Proposition

Let $\boldsymbol{A}$ be a fuzzy set defined on a finite universe and let $\boldsymbol{n}$ be a strictly decreasing negation such that $\boldsymbol{n(0.5) = 0.5}$, then

$$E(A) = \frac{\sum_{u \in \mathcal{U}} 1 - \left(A(u) - n(A(u))\right)^2}{Card(\mathcal{U})}$$

Theorem

The mapping $\boldsymbol{E}$ induced by the φ-index of inclusion and a strictly decreasing negation $\boldsymbol{n}$ such that $\boldsymbol{n}(0.5) = 0.5$, is a De Luca-Termini measure of fuzzy entropy.

Proposition

Let $\boldsymbol{A}$ be a fuzzy set defined on a finite universe and let $\boldsymbol{n}$ be a strictly decreasing negation such that $\boldsymbol{n}(0.5) = 0.5$, then

$$\boldsymbol{E}(\boldsymbol{A}) = \frac{\sum_{u \in \mathcal{U}} 1 - \big(A(u) - n(A(u))\big)^2}{Card(\mathcal{U})}$$

Theorem

Let $\mathbf{A}$ and $\mathbf{B}$ be fuzzy sets defined on a finite universe $\mathcal{U}$, then there exists an adjoint pair $(*, \rightarrow)$ such that

$$Inc(A, B)(x) = x * \left(\bigwedge_{u \in \mathcal{U}} A(u) \rightarrow B(u) \right)$$

Note the relationship with Zadeh's compositional rule of inference.

As a consequence, we obtain a Modus Ponens *à la* $\boldsymbol{f}$-index of inclusion.

$$
\begin{array}{lll}
 & \boldsymbol{A} \Rightarrow \boldsymbol{B} & \equiv \mathbf{Inc}(\boldsymbol{A};\boldsymbol{B}) \\
 & \boldsymbol{A}(\boldsymbol{u}) & \equiv \beta \\
\hline
\therefore & \boldsymbol{B}(\boldsymbol{u}) & \geq \mathbf{Inc}(\boldsymbol{A};\boldsymbol{B})(\beta)
\end{array}
$$

As a consequence, we obtain a Modus Ponens *à la* $\boldsymbol{f}$-index of inclusion.

$$
\begin{array}{lll}
 & A \Rightarrow B & \equiv \mathbf{Inc}(A;B) \\
 & A(u) & \equiv \ \beta \\
\hline
\therefore & B(u) & \geq \mathbf{Inc}(A;B)(\beta)
\end{array}
$$

- The value of the inference in terms of the $\boldsymbol{f}$-index of inclusion is the greatest among all the possible inferences that can be done using adjoint pairs.

N. Madrid and **MOA**. *The φ-index of inclusion as optimal adjoint pair for fuzzy modus ponens*. Fuzzy Sets and Systems **466**:Article 108474, 2023.

As with $\boldsymbol{f}$-inclusion, the underlying idea here is to measure contradiction not with real values, but with certain mappings in the unit interval, that should satisfy some properties that resemble negation operators.

Definition

The set of *f-indexes of contradiction* (denoted by $\bar{\Omega}$) is the set of mappings $f: [0,1] \to [0,1]$ satisfying the following properties for all $x, y \in [0,1]$:

- $f(0) = 1$;
- if $x \leq y$ then $f(y) \leq f(x)$

As with $\boldsymbol{f}$-inclusion, the underlying idea here is to measure contradiction not with real values, but with certain mappings in the unit interval, that should satisfy some properties that resemble negation operators.

Definition

The set of *$\boldsymbol{f}$-indexes of contradiction* (denoted by $\bar{\Omega}$) is the set of mappings $\boldsymbol{f} : [0, 1] \to [0, 1]$ satisfying the following properties for all $\boldsymbol{x}, \boldsymbol{y} \in [0, 1]$:

- $\boldsymbol{f}(0) = 1$;
- if $\boldsymbol{x} \leq \boldsymbol{y}$ then $\boldsymbol{f}(\boldsymbol{y}) \leq \boldsymbol{f}(\boldsymbol{x})$

Definition

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two fuzzy sets defined over a nonempty universe $\boldsymbol{\mathcal{U}}$ and let $\boldsymbol{f} \in \bar{\boldsymbol{\Omega}}$. We say that $\boldsymbol{A}$ *is* $\boldsymbol{f}$*-weakly-contradictory wrt* $\boldsymbol{B}$ if and only if $\boldsymbol{A(u)} \leq \boldsymbol{f(B(u))}$ holds for all $\boldsymbol{u} \in \boldsymbol{\mathcal{U}}$.

Definition

Let A and B be two fuzzy sets defined over a nonempty universe $\mathcal{U}$ and let $f \in \bar{\Omega}$. We say that A *is* f*-weakly-contradictory wrt* B if and only if $A(u) \leq f(B(u))$ holds for all $u \in \mathcal{U}$.

The f-weak-contradiction in [BMO15] generalizes the N-contradiction from [TAJ'99].

H. Bustince, N. Madrid, and **MOA**. *The Notion of Weak-Contradiction: Definition and Measures*. IEEE Trans. Fuzzy Systems, **23**(4) (2015), 1057–1069.

E. Trillas, C. Alsina, and J. Jacas (1999) *On contradiction in fuzzy logic*. Soft Computing, 3(4):197–199

Definition

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two fuzzy sets, the *$\boldsymbol{f}$-index of contradiction of $\boldsymbol{A}$ wrt $\boldsymbol{B}$*, denoted by $\boldsymbol{Con(A, B)}$, is given by the least mapping $\boldsymbol{f} \in \bar{\Omega}$ verifying that $\boldsymbol{A}$ is $\boldsymbol{f}$-weakly-contradictory wrt $\boldsymbol{B}$.

Definition

Let A and B be two fuzzy sets, the *f-index of contradiction of A wrt B*, denoted by $Con(A, B)$, is given by the least mapping $f \in \bar{\Omega}$ verifying that A is f-weakly-contradictory wrt B.

The following analytic expression for $Con(A, B)(x)$ can be found in [BMO15]:

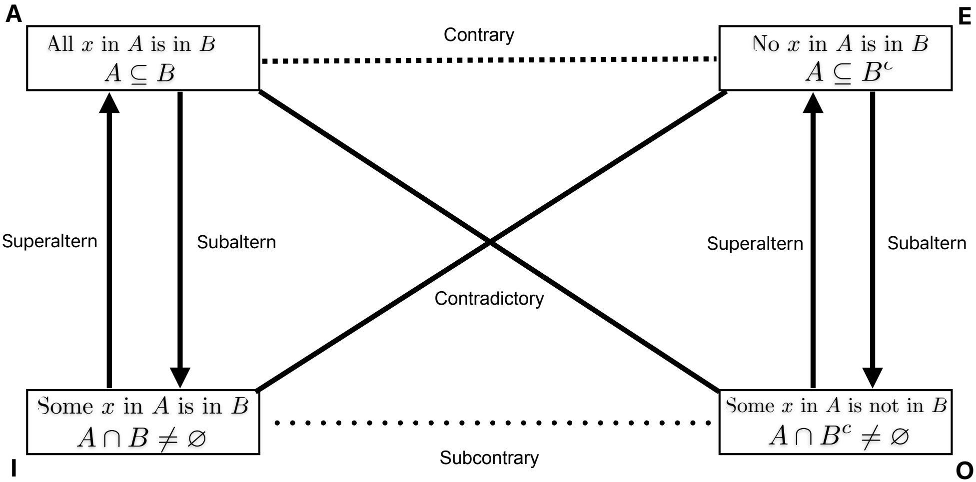
$$Con(A, B)(x) = \begin{cases} 1 & \text{if } x = 0 \\ \sup\limits_{u \in \mathcal{U}} \{A(u) \mid x \leq B(u)\} & \text{otherwise.} \end{cases}$$

H. Bustince, N. Madrid, and **MOA**. *The Notion of Weak-Contradiction: Definition and Measures*. IEEE Trans. Fuzzy Systems, **23**(4) (2015), 1057–1069.

- The $\boldsymbol{f}$-indexes of inclusion and $\boldsymbol{f}$-indexes of contradiction can be related in terms of the Aristotelian square of opposition.

Applications of f-inclusion

On the square of opposition



Applications of f-inclusion

On the square of opposition

- The $\boldsymbol{f}$-indexes of inclusion and $\boldsymbol{f}$-indexes of contradiction can be related in terms of the Aristotelian square of opposition.
- We can show that the extreme cases of the $\boldsymbol{f}$-indexes of inclusion and contradiction coincide with the vertexes of the Aristotelian square of opposition.

- The $\boldsymbol{f}$-indexes of inclusion and $\boldsymbol{f}$-indexes of contradiction can be related in terms of the Aristotelian square of opposition.
- We can show that the extreme cases of the $\boldsymbol{f}$-indexes of inclusion and contradiction coincide with the vertexes of the Aristotelian square of opposition.
- The rest of $\boldsymbol{f}$-indexes can be allocated in the diagonals of the extreme cases.

- The $\boldsymbol{f}$-indexes of inclusion and $\boldsymbol{f}$-indexes of contradiction can be related in terms of the Aristotelian square of opposition.
- We can show that the extreme cases of the $\boldsymbol{f}$-indexes of inclusion and contradiction coincide with the vertexes of the Aristotelian square of opposition.
- The rest of $\boldsymbol{f}$-indexes can be allocated in the diagonals of the extreme cases.
- We prove that the Contradiction, Contrariety, Subcontrariety, Subalternation and Superalternation relations also hold between the $\boldsymbol{f}$-indexes of inclusion and contradiction.

Applications of f-inclusion

On the square of opposition

- The $\boldsymbol{f}$-indexes of inclusion and $\boldsymbol{f}$-indexes of contradiction can be related in terms of the Aristotelian square of opposition.
- We can show that the extreme cases of the $\boldsymbol{f}$-indexes of inclusion and contradiction coincide with the vertexes of the Aristotelian square of opposition.
- The rest of $\boldsymbol{f}$-indexes can be allocated in the diagonals of the extreme cases.
- We prove that the Contradiction, Contrariety, Subcontrariety, Subalternation and Superalternation relations also hold between the $\boldsymbol{f}$-indexes of inclusion and contradiction.

N. Madrid and **MOA**. *Approaching the square of oppositions in terms of the φ-indexes of inclusion and contradiction*. Under review (R1 Submitted)

Proposition (Contrariety and Subcontrariety)

Let A and B be two fuzzy sets such that A is normal, then $Inc(A, B) < Con(B, A)$.

Proposition (Subaltern f-index of inclusion)

Let A and B be two fuzzy sets such that A is normal, then there exists $u \in \mathcal{U}$ such that $(A \cap B)(u) \geq Inc(A, B)(1)$.

Proposition (Subaltern f-index of contradiction)

Let A and B be two fuzzy sets such that A is normal and let n be an involutive negation. Then there exists $u \in \mathcal{U}$ such that $(A \cap B^c)(u) \geq n(Con(B, A)(1))$.

- Recently we have focused on providing a sound and complete logic to formalise and reason about $\boldsymbol{f}$-indices of inclusion.
- That logic is based on the S5-like modal extension of $\boldsymbol{n}$-valued Łukasiewicz logic with truth-constants, $\boldsymbol{S5}(Ł_n^c)$.
- We take advantage of the good logical and expressive properties of this logical setting to define the logic $IŁ_n$ to *reason* about $\boldsymbol{f}$-indexes of inclusion.
- We provide a first step towards a logical account of the notion of $\boldsymbol{f}$-index of inclusion for fuzzy sets. We plan to consider a more general many-valued logical setting lifting the assumption of dealing with finitely-many truth-degrees.

T. Flaminio, Ll. Godo, N. Madrid, **MOA**. *A logic to reason about $\boldsymbol{f}$-indices of inclusion over $Ł_n$*. Proc. of the Eur. Conf. on Fuzzy Logic and Technology (EUSFLAT), 2023.

Conclusions and future work

- We have seen that strong indications that the $\boldsymbol{f}$-index of inclusion is a convenient operator to represent inclusion between fuzzy sets.
- Currently, we are working on the integration of the $\boldsymbol{f}$-index of inclusion with fuzzy formal concept analysis.

Closing

Thank you for your attention!!

And these are the curious things I told you I had to tell, and if you don't think them so, I have got no others.

Y estas son las maravillas que dije que os había de contar.
Y si no os lo han parecido, no sé otras.

[[Quijote II, cap 25]]

Don Quixote

Int Conf on Conceptual Structures

Call for participation ICCS'23 https://iccs-conference.org/

ICCS

- **When?** September 11 – 13, 2023,
- **Where?** Humboldt-Universität zu Berlin, Germany
- **Why?** Interesting program, no registration fee, … Why not?
- **Keynote Speakers**
    - Nina Gierasimczuk: *The Dynamics of True Belief – Learning by Revision and Merge*
    - Henrik Müller: *What's in a story? How narratives structure the way we think about the economy*
    - Camille Roth: *Semantic graphs and social networks*

On the φ-degree of inclusion

Manuel Ojeda-Aciego

Departamento Matemática Aplicada

Universidad de Málaga (Spain)

(mainly based on joint work with Nicolás Madrid)

July 20, 2023

ICFCA, Kassel